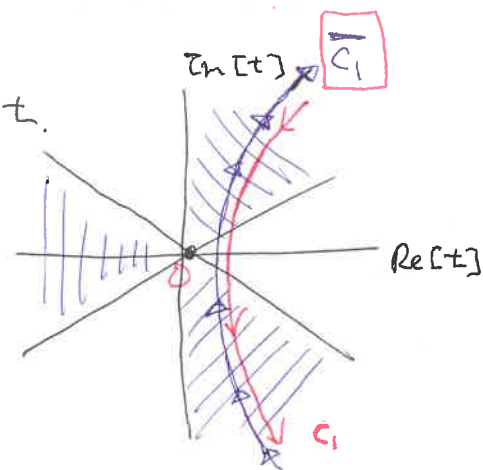


② $z \rightarrow -\infty$: Let $S = -|z|^{\frac{3}{2}} t$ (inversion w.r.t. 0)

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$$Ai(z) = -\frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{C_1} e^{|z|^{\frac{3}{2}} (t + \frac{t^3}{3})} dt$$

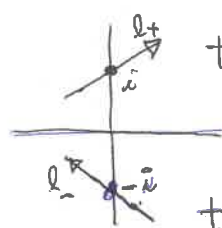
$$= \frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{\bar{C}_1} e^{|z|^{\frac{3}{2}} (t + \frac{t^3}{3})} dt$$



→ Saddle point approx.

(two points : $t = \pm \tilde{u}$
equivalent)

→ Deform \bar{C}_1 to cross $t = \pm \tilde{u}$



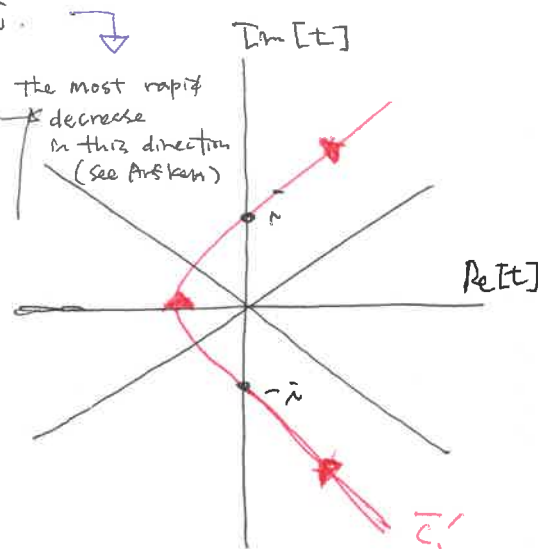
$$t = \tilde{u} + \frac{2}{3}\sqrt{\tilde{u}}$$

$$\|\sqrt{\tilde{u}}\| = e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}$$

the most rapid decrease in this direction (see Arstein)

$$t = -\tilde{u} - \frac{2}{3}\sqrt{-\tilde{u}}$$

$$\|\sqrt{-\tilde{u}}\| = e^{-\frac{\pi i}{4}} = \frac{1-i}{\sqrt{2}}$$



$$t + \frac{t^3}{3} \approx \frac{2}{3}\tilde{u} - \frac{2}{3}\tilde{u}^2 \text{ around } t = +\tilde{u}$$

$$\approx -\frac{2}{3}\tilde{u} - \frac{2}{3}\tilde{u}^2 \text{ around } t = -\tilde{u}$$

$$Ai(z) \approx \frac{1}{2\pi i} |z|^{\frac{1}{2}} \left[e^{\frac{\pi i}{4}\tilde{u}} \int_{l_+} d\tilde{z} e^{|z|^{\frac{3}{2}} (\frac{2}{3}\tilde{u} - \frac{2}{3}\tilde{z}^2)} - e^{-\frac{\pi i}{4}\tilde{u}} \int_{l_-} d\tilde{z} e^{|z|^{\frac{3}{2}} (-\frac{2}{3}\tilde{u} - \frac{2}{3}\tilde{z}^2)} \right]$$

approx. $\int_{l_{\pm}} d\tilde{z} \rightarrow \int_{-\infty}^{\infty} d\tilde{z}$ (integrand is only important near $\tilde{z} = 0$)

$$\approx \frac{1}{2\pi i} |z|^{-\frac{1}{4}} \left[e^{i(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4})} - e^{-i(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4})} \right]$$

$$= \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin \left[\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4} \right]$$

or

$$Ai(z) \approx \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \cos \left[\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4} \right] \text{ as } z \rightarrow -\infty$$

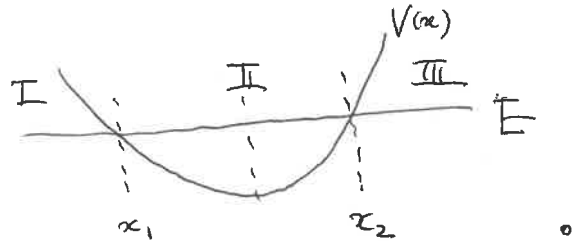
Similarly,

$$Bi(z) \approx \begin{cases} \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} e^{\frac{2}{3} |z|^{\frac{3}{2}}} & \text{as } z \rightarrow \infty \\ -\frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left(\frac{2}{3} |z|^{\frac{3}{2}} - \frac{\pi}{4}\right) & \text{as } z \rightarrow -\infty \end{cases}$$

← unphysical!

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Now, coming back to



$$I: u_E(x) \sim \exp\left[-\frac{1}{\hbar} \int_x^{x_1} dx' \sqrt{2m[V(x') - E]}\right]$$

near x_1 , $V(x) \simeq E + |V'(x_1)| (x_1 - x) + \dots$

$$\Rightarrow \frac{1}{\hbar} \int_x^{x_1} dx' \sqrt{2m(x_1 - x') |V'(x_1)|} = \sqrt{\frac{2m |V'|}{\hbar^2}} \cdot \frac{2}{3} (x_1 - x)^{\frac{3}{2}}$$

$$= \frac{2}{3} z^{\frac{3}{2}} \quad \parallel \quad z = \left(\frac{2m |V'|}{\hbar^2}\right)^{\frac{1}{3}} (x_1 - x)$$

$$\therefore u_E(x) \sim A_0^*(z) \quad \text{as } z \rightarrow \infty$$

$$III: u_E(x) \sim \exp\left[-\frac{1}{\hbar} \int_{x_2}^x dx' \sqrt{2m[V(x') - E]}\right]$$

near x_2 , $V(x) \simeq E + |V'(x_2)| (x - x_2) + \dots$

$$\Rightarrow \frac{1}{\hbar} \int_{x_2}^x dx' \sqrt{2m |V'(x_2)| (x' - x_2)} = \sqrt{\frac{2m |V'|}{\hbar^2}} \cdot \frac{2}{3} (x - x_2)^{\frac{3}{2}}$$

$$= \frac{2}{3} z^{\frac{3}{2}} \quad \parallel \quad z = \left(\frac{2m |V'|}{\hbar^2}\right)^{\frac{1}{3}} (x - x_2)$$

$$\therefore u_E(x) \sim A_i(z) \quad \text{as } z \rightarrow \infty$$

II : ① from x_1

$$U_E(x) \sim \cos \left[\frac{1}{\hbar} \int_{x_1}^x dx' \sqrt{2m(E-V(x'))} - \frac{\pi}{4} \right]$$

$$\text{def. of } z \text{ in I} \rightarrow \frac{2}{3} |z|^{\frac{3}{2}}$$

from $A_i(z)$ as $z \rightarrow -\infty$

② from x_2

$$U_E(x) \sim \cos \left[\frac{1}{\hbar} \int_x^{x_2} dx' \sqrt{2m(E-V(x'))} - \frac{\pi}{4} \right]$$

$$\text{def. of } z \text{ in III} \rightarrow \frac{2}{3} |z|^{\frac{3}{2}}$$

from $A_i(z)$ as $z \rightarrow -\infty$

Phase Matching For ALL x

$$\frac{1}{\hbar} \int_{x_1}^x dx' \left[\right] - \frac{\pi}{4} = - \frac{1}{\hbar} \int_x^{x_2} dx' \left[\right] + \frac{\pi}{4} + n\pi$$

$\Leftrightarrow \cos \theta = \cos(-\theta)$

\Rightarrow Quantization condition (WKB)

$$\int_{x_1}^{x_2} dx' \sqrt{2m(E-V(x'))} = \left(n + \frac{1}{2} \right) \pi \hbar$$

\Leftrightarrow Sommerfeld - Wilson quantization
(old quantum theory)

$$\oint p dx = n h$$

$$\parallel p_{cl} = \sqrt{2m(E-V)}$$

• Why is it "semi-classical" ?

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(MORE RIGOROUSLY...)

Hamilton - Jacobi Theory

Classical Action $S(b, a) = \int_{t_a}^{t_b} dt \mathcal{L}(q(t), \dot{q}(t), t)$

Where $b \equiv q_b \equiv (q_1(t_b), q_2(t_b), \dots, q_f(t_b))$

$a \equiv q_a \equiv (q_1(t_a), q_2(t_a), \dots, q_f(t_a))$

def. Integral of \mathcal{L} along the trajectory

allowed by the eqs. of motion It's a physical trajectory.

from configuration q_a at t_a to q_b at t_b .

Taking an arbitrary variation δq ?

$$\delta S(b, a) = \int_{t_a}^{t_b} dt \left(\frac{\partial \mathcal{L}}{\partial q_{\tilde{i}}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_{\tilde{i}}} \right) \delta q_{\tilde{i}} \quad \xrightarrow{=0} \text{We're on a physical trajectory!}$$

* NOTE !

$$\int_t \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} dt$$

\hookrightarrow int. by parts

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big|_t - \int_t \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q dt$$

$$+ \frac{\partial \mathcal{L}}{\partial q_{\tilde{i}}} \delta q_{\tilde{i}} \Big|_{t_a}^{t_b}$$

$\neq 0$: We do have variations at the end points !

$$\Rightarrow \delta S(b, a) = P_{b, \tilde{i}} \delta q_{\tilde{i}}(t_b) - P_{a, \tilde{i}} \delta q_{\tilde{i}}(t_a)$$

$$\Rightarrow P_{b, \tilde{i}} = \frac{\partial S(b, a)}{\partial q_{b, \tilde{i}}} \quad , \quad P_{a, \tilde{i}} = - \frac{\partial S(b, a)}{\partial q_{a, \tilde{i}}} \\ \equiv P_{b, \tilde{i}}(b, a) \quad \quad \quad \equiv P_{a, \tilde{i}}(b, a)$$

Fix (q_a, t_a) ,

$$S(b, a) = \int_{t_a}^{t_b} dt L(q, \dot{q}; t)$$

Consider the change of action in t_b

$$\Rightarrow \frac{dS}{dt_b} = L(t_b)$$

$$L \Rightarrow \frac{\partial S}{\partial t_b} + \frac{\partial S}{\partial q_{b,i}} \dot{q}_{b,i} = \frac{\partial S}{\partial t_b} + p_{b,i} \dot{q}_{b,i}$$

$$\text{Thus, } \frac{\partial S}{\partial t_b} + \underbrace{p_{b,i} \dot{q}_{b,i}}_{\equiv H(t_b)} - L(t_b) = 0$$

$$\text{Then } S \equiv S(q_1, \dots, q_f, t) \quad \begin{matrix} \downarrow t \equiv t_b \\ q = q_b \end{matrix}$$

$$\text{for } H \equiv H(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}; t).$$

$$\Rightarrow \boxed{\frac{\partial S}{\partial t} + H = 0} \Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V(q) = 0$$

Hamilton - Jacobi equation

• For a time-indep. Hamiltonian, ($H = E$)

$$S(q, t) = W(q) - Et \quad \left(\begin{matrix} \text{integrate} \\ \frac{\partial S}{\partial t} + H = 0 \end{matrix} \right)$$

↑ Hamilton's characteristic function.

• If we consider a free particle in a potential

$$|\nabla S| = |\nabla W| = \sqrt{2m(E - V)}$$

\equiv momentum.

1D: If $V=0$,

$$W(x) \sim px$$

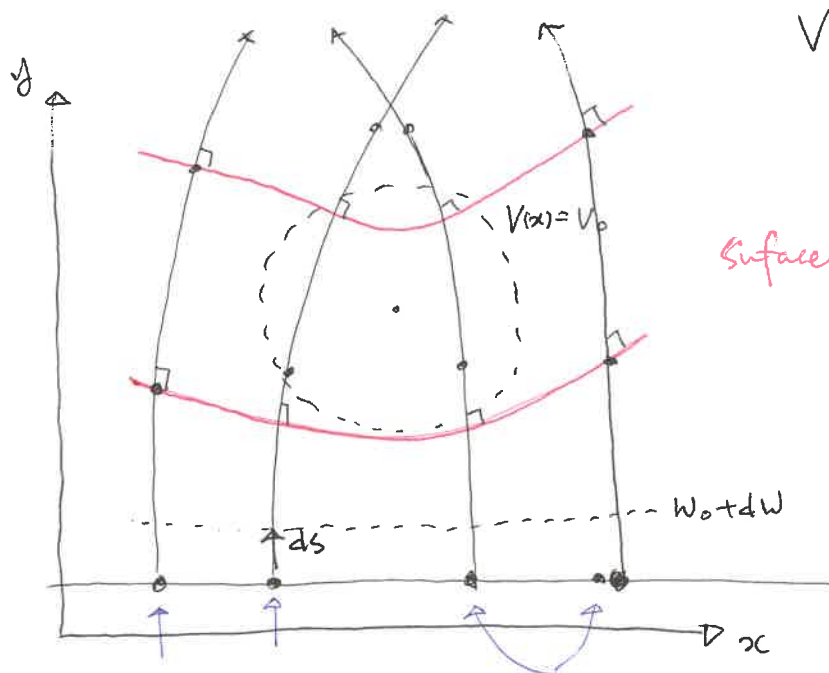
$$S(x, t) \sim px - Et$$

* meaning of S and W .

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consider a particle moving in a potential

$$V(x) = -\frac{1}{1+x^2+y^2}$$



Surfaces of $W(q) = \text{const}$
(equi- W planes)

$$W(q) = W_0 \text{ at } t = t_0$$

different initial positions.

Along the trajectory, length ds for $W_0 \rightarrow W_0 + dW$

$$\Rightarrow ds = \frac{dW}{|\nabla W|} = \frac{dW}{\sqrt{2m[E - V_q]}} \quad (V_q \equiv V(q))$$

The motion of $S(q, t) = \text{constant}$, $(dW = E dt)$
 $\therefore ds = 0$.

$$\Rightarrow \frac{ds}{dt} = \frac{E}{\sqrt{2m[E - V_q]}}$$

It's like
 $S \sim px - Et$

If the surfaces of $S = \text{const.}$ as wave-fronts,

$$\frac{ds}{dt} = \underline{\text{phase velocity}}$$

[ex. $e^{i(hx - \hbar\omega t)}$]
 $\rightarrow v = \frac{dx}{dt} = \frac{\omega}{k}$

\Rightarrow Classical Mechanics

(particles are not
on the surface
(wave fronts))

\approx Geometrical optics

→ Semi classical Interpretation of the Wave function

- From Hamilton-Jacobi Theory (C.M.) (Q.M.).

$S \approx$ phase factor of wave fn.

- Similarity to WKB approx.

HJ eq. for $H = \frac{p^2}{2m} + V(x)$ (t-indep.)

$$\frac{\partial S}{\partial t} + H = 0 \quad \text{with} \quad S = W(x) - Et$$

$$\Rightarrow -E + \frac{1}{2m} \left(\frac{\partial W}{\partial x} \right)^2 + V(x) = 0.$$

... HJ eq.

$$\hookrightarrow - \left(\frac{\partial W}{\partial x} \right)^2 + \left[\frac{2m}{\hbar^2} (E - V(x)) \right] \cdot \hbar^2 = 0 \quad (\text{C.M.})$$

→ This is just WKB!

where $u_E(x) = e^{iW(x)/\hbar}$

Q.M.

⇒ Brillouin-Wentzel Ansatz

$$\Psi(x, t) = \exp \left[i \underline{\Phi(x, t) / \hbar} \right]$$

→ S

Schrödinger eq.

→

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2m} \left(\frac{\partial \Phi}{\partial x} \right)^2 + V(x) = \frac{\hbar}{2m} \frac{\partial^2 \Phi}{\partial x^2}$$

\hbar is here only.

$$\left(\frac{\partial S}{\partial x} \right)^2 \gg \hbar \left| \frac{\partial^2 S}{\partial x^2} \right|$$

as in

WKB.

Without this term, the Schrödinger eq.

is just the HJ eq.

Thus, we may just write down

$$\Psi \sim \exp \left[\frac{i}{\hbar} \cdot \text{Classical Action} \right] \text{ when } \hbar \rightarrow 0.$$

To see the quantum corrections, consider

$$\hat{H} = \underbrace{S}_{\text{classical}} + \frac{\hbar}{i} S_1 + \left(\frac{\hbar^2}{i^2} \right)^2 S_2 + \dots$$

$$\Rightarrow \mathcal{O}(\hbar^0): \quad \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V(x) = 0$$

H-J eq.

$$\mathcal{O}(\hbar^1): \quad \frac{\partial S_1}{\partial t} + \frac{1}{2m} \left[2 \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial S_1}{\partial x} + \frac{\partial^2 S}{\partial x^2} \right] = 0.$$

\rightarrow Find S_1 for a given S .

\Rightarrow Semi classical wave function

$$\Psi_{sc}(x,t) = \sqrt{\rho(x,t)} e^{i \frac{S(x,t)}{\hbar}}$$

where

$$\rho(x,t) = |\Psi_{sc}|^2 = e^{2S_1(x,t)}$$

\rightarrow prob. density of finding a particle

near x at t .

(Max Born)

* Remark

"classical" means
here when $\hat{H} \rightarrow S$.

for a very small S_1 ,
H-J eq. is recovered

when

$$\left(\frac{\partial S}{\partial x} \right)^2 \gg \hbar \left(\frac{\partial^2 S}{\partial x^2} \right).$$

i short wave length

(see WKB)

Rewriting $O(\hbar')$ terms for $\rho = e^{2S}$,

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$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[\frac{1}{m} \cdot \frac{\partial S}{\partial x} \cdot \rho(x,t) \right] = 0.$$

using the current density $\vec{J}(\vec{x}, t)$

$$\vec{J}(\vec{x}, t) = \frac{\hbar}{m} \operatorname{Im} [\psi^* \nabla \psi] = \frac{\rho \cdot \nabla S}{m}$$

$\hookrightarrow \nabla \rho \cdot \nabla \rho + \frac{\hbar}{m} \rho \nabla S$

Thus, $O(\hbar')$ terms give.

$$\int d^3 \vec{J} = \frac{\langle \vec{P} \rangle}{m}$$

(a single particle!)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad \text{continuity eq.}$$

prob. is conserved!
flux

It just behaves
like a "flux".

You can directly show this

from the Schrödinger eq, also.

Hamilton - Jacobi equation
particle

\updownarrow as $\hbar \rightarrow 0$, short wavelength

Schrödinger equation

wave

particle-wave duality

wave

"geometrical optics"

conceptually ...